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A CHARACTERIZATION OF THE VALUE OF ZERO-SUM TWO-PERSON GAMES

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ABSTRACT

For the family D, consisting of those zero-sum two-person games which have a value, the value-function on D is characterized by four properties called objectivity, monotony, symmetry and sufficiency.

INTRODUCTION

In a beautiful paper, E. I. Vilkas gave a characterization of the value-function, defined on the class of all finite matrix games [2]. In [1], pp. 60-65 this result was extended to the class of all finite and semi-infinite matrix games.

The purpose of this paper is to deduce characterizing properties for the value-function on the set of all determined two-person games. The organization of the paper is as follows: the necessary notation and definitions are given in sections 1 and 2; in section 3, properties for the value-function are presented, which are shown in section 4 to be characteristic of this function.

1. A (*zero-sum*) *two-person game* is an ordered triple $<X,Y,K>$, in which X and Y are nonempty sets (called the *pure strategy spaces* of player I and player II, respectively) and $K : X \times Y \rightarrow \mathbb{R}$ is a real-valued function on the Cartesian product of X and Y (called the *pay off function* of player I).

2. Let $<X,Y,K>$ be a two-person game. For each $x \in X$ ($y \in Y$) let us denote the probability measure on X (Y) with mass 1 in x (y) by e_x (e_y). Let P_X be the set of all convex combinations of elements of $\{e_x;\ x \in X\}$; likewise let P_Y be the convex hull of $\{e_y;\ y \in Y\}$. Then the two-person game $<P_X,\ P_Y, E_K>$ with

$$E_K(\mu,\nu) := \int\int K(x,y)\ d\mu(x)\ d\nu(y) \text{ for each } (\mu,\nu) \in P_X \times P_Y$$

is called the *c-mixed extension of the game* $<X,Y,K>$. The *lower value* $\sup_{\mu \in P_X} \inf_{\nu \in P_Y} E_K(\mu,\nu)$ of the game $<P_X,P_Y,E_K>$ is denoted by $\underline{v}(X,Y,K)$ and the *upper value* $\inf_{\nu \in P_Y} \sup_{\mu \in P_X} E_K(\mu,\nu)$ is denoted by $\overline{v}(X,Y,K)$. Note that

$$-\infty \leqslant \underline{v}(X,Y,K) \leqslant \overline{v}(X,Y,K) \leqslant \infty.$$

If $\underline{v}(X,Y,K) = \overline{v}(X,Y,K)$ for a game $<X,Y,K>$, then we say that the game is a *determined game*. In that case, the common value is denoted by $v(X,Y,K)$ and called the *value* of (the c-mixed extension of) $<X,Y,K>$. The family of determined games is denoted by D.

3. In this section we want to look at some distinguished properties of the value-function $v : D \to [-\infty, \infty]$. For this purpose we need some definitions.

DEFINITION 1: The *transpose* of a two-person game $<X, Y, K>$ is the two-person game $<Y, X, -K'>$ where

$$K'(y,x) := K(x,y) \text{ for each } (y,x) \in Y \times X.$$

DEFINITION 2: Let $<X, Y, K>$ be a two-person game and let S be a nonempty subset of X. Then we say that S is *sufficient for player I in the game* $<X, Y, K>$ if for each $x \in X - S$ there exists a $\mu \in P_S$ such that

$$E_K(\mu, e_y) \geqslant K(x,y) \text{ for each } y \in Y.$$

DEFINITION 3: Let $<X, Y, K>$ be a two-person game and let T be a nonempty subset of Y. We say that T *is sufficient for player II in the game* $<X, Y, K>$ if T is sufficient for player I in the game $<Y, X, -K'>$.

THEOREM 1:

(P.1) ["Objectivity"] Let $<X, Y, K>$ be a two-person game and suppose that $X = \{a\}$, $Y = \{b\}$. Then $<X, Y, K> \in D$ and $v(X, Y, K) = K(a,b)$.

(P.2) ["Monotonicity"] Let $<X, Y, K> \in D$ and $<X, Y, L> \in D$ and suppose that $L \geqslant K$ (i.e. $L(x,y) \geqslant K(x,y)$ for each $(x,y) \in X \times Y$). Then $v(X, Y, L) \geqslant v(X, Y, K)$.

(P.3) ["Symmetry"] Let $<X, Y, K> \in D$. Then $<Y, X, -K'> \in D$ and $v(Y, X, -K') = -v(X, Y, K)$.

(P.4) ["Sufficiency"] Let $<X, Y, K>$ be a two-person game, and $\phi \neq S \subset X$ and let $K' : S \times Y \to \mathbb{R}$ be the restriction of K to $S \times Y$. Suppose that S is sufficient for player I in the game $<X, Y, K>$. Then $<S, Y, K'> \in D$ iff $<X, Y, K> \in D$ and

$$v(X, Y, K) = v(S, Y, K') \text{ if } <S, Y, K'> \in D.$$

PROOF: (P.1) and (P.2) are obvious. (P.3) follows from the fact that

$$-E_K(\mu, \nu) = E_{-K'}(\nu, \mu) \text{ for each } (\mu, \nu) \in P_X \times P_Y.$$

Now let us prove (P.4). First we note that P_S can be seen (in an obvious manner) as a subset of P_X, and that $E_{K'}$ is the restriction of E_K to $P_S \times P_Y$.

Take $\alpha \in P_X$. Then there exist $n \in \mathbb{N}$, $x^1, x^2, \ldots, x^n \in X$ and $p_1, p_2, \ldots, p_n \in [0, \infty)$ such that $\sum_{i=1}^{n} p_i = 1$ and $\alpha = \sum_{i=1}^{n} p_i e_{x^i}$. Since S is sufficient for player I in the game $<X, Y, K>$ for each $i \in \{1, \ldots, n\}$, there exists an $\alpha_i \in P_S$ such that

$$E_K(\alpha_i, e_y) \geqslant K(x^i, y) \text{ for each } y \in Y.$$

[If $x^i \in S$, then we can take $\alpha_i = e_{x^i}$.] Let $\bar{\alpha} := \sum_{i=1}^{n} p_i \alpha_i$. Then $\bar{\alpha} \in P_S$ and

$$E_{K'}(\bar{\alpha}, e_y) = \sum_{i=1}^{n} p_i E_K(\alpha_i, e_y) \geqslant \sum_{i=1}^{n} p_i K(x^i, y) = E_K(\alpha, e_y) \text{for each } y \in Y.$$

But then,

(i) $\quad E_{K'}(\bar{\alpha},\nu) \geqslant E_K(\alpha,\nu)$ for each $\alpha \in P_X$ and each $\nu \in P_Y$.

This implies that

$$\sup_{\mu \in P_S} E_{K'}(\mu,\nu) = \sup_{\alpha \in P_X} E_K(\alpha,\nu) \text{ for each } \nu \in P_Y$$

and thus

(ii) $\quad \bar{v}(S,Y,K') = \bar{v}(X,Y,K)$.

From (i) we may also conclude that

$$\inf_{\nu \in P_Y} E_{K'}(\bar{\alpha},\nu) \geqslant \inf_{\nu \in P_Y} E_K(\alpha,\nu) \text{ for each } \alpha \in P_X$$

and then

(iii) $\quad \underline{v}(S,Y,K') = \underline{v}(X,Y,K)$.

Now (P.4) follows from (ii) and (iii). ||

4. The following theorem shows that the properties (P.1)-(P.4) characterize the value-function $v : D \to [-\infty,\infty]$.

THEOREM 2: Let $f : D \to [-\infty,\infty]$ be a function with the following four properties:

(Q.1) If $X = \{a\}$, $Y = \{b\}$ and if K is a real-valued function on $X \times Y$, then $f(X,Y,K) = K(a,b)$.

(Q.2) For each $<X,Y,K> \in D$, $<X,Y,L> \in D$ with $L \geqslant K : f(X,Y,L) \geqslant f(X,Y,K)$.

(Q.3) For each $<X,Y,K> \in D : f(Y,X,-K') = -f(X,Y,K)$.

(Q.4) For each $<X,Y,K> \in D$ and $<S,Y,K'> \in D$, where $S \subset X$, K' is the restriction of K to $S \times Y$ and where S is sufficient for player I in the game $<X,Y,K>$, we have $f(S,Y,K') = f(X,Y,K)$.

Then $f(X,Y,K) = v(X,Y,K)$ for each $<X,Y,K> \in D$.

PROOF: First we note that (Q.3) and (Q.4) imply

(Q.5) For each $<X,Y,K> \in D$ and $<X,T,K''> \in D$, where $T \subset Y$, K'' is the restriction of K to $X \times T$ and where T is sufficient for player II in the game $<X,Y,K>$, we have $f(X,T,K'') = f(X,Y,K)$.

Now take an $<X,Y,K> \in D$ with $v(X,Y,K) \in (-\infty,\infty]$ and take a real number t such that $v(X,Y,K) > t$. We want to prove that $f(X,Y,K) \geqslant t$. For this purpose we introduce the following five two-person games:

(1) $<X \cup \{a\}, Y, L>$ where $a \notin X$ and where $L(x,y) := K(x,y)$ for each $(x,y) \in X \times Y$ and $L(a,y) := t$ for each $y \in Y$.

(2) $<X \cup \{a\}, Y, M>$ where $M(x,y) :=$ minimum $\{K(x,y), t\}$ for each $(x,y) \in (X \cup \{a\}) \times Y$.

(3) $<X \cup \{a\}, Y \cup \{b\}, N>$ where $b \notin Y$ and where $N(x,y) := M(x,y)$ for each $(x,y) \in (X \cup \{a\}) \times Y$ and $N(x,b) := t$ for each $x \in X \cup \{a\}$.

(4) $<\{a\}, Y \cup \{b\}, N'>$ where N' is the restriction of N to $\{a\} \times (Y \cup \{b\})$.

(5) $<\{a\}, \{b\}, N''>$ where N'' is the restriction of N' to $\{a\} \times \{b\}$.

Since $\mathrm{v}(X,Y,K) > t$, there exists a $\mu \in P_X$ such that

$$E_L(\mu, e_y) = E_K(\mu, e_y) \geqslant t = L(a,y) \text{ for each } y \in Y.$$

Hence, X is sufficient for player I in the game $<X \cup \{a\}, Y, L>$. By (P.4) and (Q.4) we may conclude that

(Q.6) $<X \cup \{a\}, Y, L> \in D$ and $f(X \cup \{a\}, Y, L) = f(X,Y,K)$.

It follows from (Q.1) that

(Q.7) $f(\{a\}, \{b\}, N'') = t$.

In the game $<\{a\}, Y \cup \{b\}, N'>$ the set $\{b\}$ is sufficient for player II because

$$E_{N'}(e_a, e_b) = N'(a,y) = t \text{ for each } y \in Y.$$

Then $<\{a\}, Y \cup \{b\}, N'> \in D$ in view of (P.3) and (P.4); now (Q.5) implies

(Q.8) $f(\{a\}, Y \cup \{b\}, N') = f(\{a\}, \{b\}, N'')$.

In the game $<X \cup \{a\}, Y \cup \{b\}, N>$ the set $\{a\}$ is sufficient for player I because for each $x \in X$:

$$t = E_N(e_a, e_y) \geqslant N(x,y) \text{ for each } y \in Y \cup \{b\}.$$

By (P.4) and (Q.4) we obtain: $<X \cup \{a\}, Y \cup \{b\}, N> \in D$ and

(Q.9) $f(X \cup \{a\}, Y \cup \{b\}, N) = f(\{a\}, Y \cup \{b\}, N')$.

It is easy to see that Y is sufficient for player II in the game $<X \cup \{a\}, Y \cup \{b\}, N>$. Hence, by (P.3) and (P.4): $<X \cup \{a\}, Y, M> \in D$; and then by (Q.5):

(Q.10) $f(X \cup \{a\}, Y, M) = f(X \cup \{a\}, Y \cup \{b\}, N)$.

Now $L \geqslant M$ and then by (Q.2) we have

(Q.11) $f(X \cup \{a\}, Y, L) \geqslant f(X \cup \{a\}, Y, M)$.

Combining (Q.6)-(Q.11) we obtain: $f(X,Y,K) \geqslant t$. Thus, we have proved that $f(X,Y,K) \geqslant t$ for each $<X,Y,K> \in D$ with $\mathrm{v}(X,Y,K) \in (-\infty, \infty]$ and each $t < \mathrm{v}(X,Y,K)$. But then

(Q.12) $f(X,Y,K) \geqslant \mathrm{v}(X,Y,K)$ for each $<X,Y,K> \in D$.

It follows from (Q.3), (Q.12) and (P.3) that

(Q.13) $f(X,Y,K) = -f(Y,X,K') \leqslant -\mathrm{v}(Y,X,K') = \mathrm{v}(X,Y,K)$ for each $<X,Y,K> \in D$.

Properties (Q.12) and (Q13) imply the conclusion of the theorem. $||$

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